

# Bounded and Semibounded Representations of Infinite Dimensional Lie Groups

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**Abstract.** In this note we describe the recent progress in the classification of bounded and semibounded representations of infinite dimensional Lie groups. We start with a discussion of the semiboundedness condition and how the new concept of a smoothing operator can be used to construct  $C^*$ -algebras (so called host algebras) whose representations are in one-to-one correspondence with certain semibounded representations of an infinite dimensional Lie group  $G$ . This makes the full power of  $C^*$ -theory available in this context.

Then we discuss the classification of bounded representations of several types of unitary groups on Hilbert spaces and of gauge groups. After explaining the method of holomorphic induction as a means to pass from bounded representations to semibounded ones, we describe the classification of semibounded representations for hermitian Lie groups of operators, loop groups (with infinite dimensional targets), the Virasoro group and certain infinite dimensional oscillator groups.

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## 1. Introduction

This article surveys some of the recent developments in the representation theory of infinite dimensional Lie groups. Here an infinite dimensional Lie group is a group  $G$  which is a smooth manifold modeled on a locally convex space for which the group operations are smooth. As usual, we identify the Lie algebra  $\mathfrak{g}$  of  $G$  with the tangent space  $T_1(G)$  in the neutral element  $\mathbf{1}$ . We shall assume throughout that  $G$  has an *exponential function*, i.e., a smooth map  $\exp: \mathfrak{g} \rightarrow G$  such that, for every  $x \in \mathfrak{g}$ , the curve  $\gamma_x(t) := \exp(tx)$  is a one-parameter group with  $\gamma'_x(0) = x$ . We refer to [31] for the basics of infinite dimensional Lie theory.

Typical examples of infinite dimensional Lie groups are

- *Banach-Lie groups*, such as the group  $\mathrm{GL}(E)$  of invertible bounded linear operators on a Banach space  $E$ , or, more generally, the group  $\mathcal{A}^\times$  of units in a unital Banach algebra  $\mathcal{A}$ . For  $G = \mathcal{A}^\times$ , the Lie algebra  $\mathfrak{g}$  can be identified with  $\mathcal{A}$ , endowed with the commutator bracket, and the exponential function is given by the convergent exponential series  $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ .

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- *Mapping groups*, such as the group  $G = C_c^\infty(M, K)$  of compactly supported smooth functions on a finite dimensional  $\sigma$ -compact manifold with values in a Lie group  $K$ .<sup>1</sup> Then  $\mathfrak{g} = C_c^\infty(M, \mathfrak{k})$ , with the Lie bracket  $[\xi, \eta](m) := [\xi(m), \eta(m)]$ , and the exponential function is given by  $\exp(\xi) := \exp_K \circ \xi$  for  $\xi \in \mathfrak{g}$ .
- *Diffeomorphism groups*, such as the group  $\text{Diff}_c(M)$  of compactly supported diffeomorphisms of a finite dimensional manifold. Then the Lie algebra is the space  $\mathcal{V}_c(M)$  of compactly supported smooth vector fields and the exponential function is given by the time-1-flow map  $\exp(X) = \Phi_1^X$ , where  $(t, m) \mapsto \Phi_t^X(m)$  is the flow of the vector field  $X$ .

We are interested in *unitary representations*, i.e., homomorphisms  $\pi: G \rightarrow \text{U}(\mathcal{H})$  into the unitary group of a complex Hilbert space  $\mathcal{H}$  for which all orbit maps  $\pi^v: G \rightarrow \mathcal{H}, g \mapsto \pi(g)v$  are continuous. The exponential function permits us to associate to every unitary representation  $(\pi, \mathcal{H})$  and every  $x \in \mathfrak{g}$  the unitary one-parameter group  $\pi_x(t) := \pi(\exp tx)$ . By Stone's Theorem, there exists a selfadjoint operator  $A := -i\overline{d\pi}(x)$  on  $\mathcal{H}$  with

$$\pi_x(t) = \pi(\exp tx) = e^{t\overline{d\pi}(x)} = e^{itA} \quad \text{for } t \in \mathbb{R}, \quad (1)$$

where the exponential on the right is to be understood in the sense of functional calculus of selfadjoint operators.

To develop a reasonably general theory of unitary representations of infinite dimensional Lie groups, new approaches have to be developed which neither rest on the fine structure theory available for finite dimensional groups nor on the existence of invariant measures. One thus has to specify suitable classes of representations for which it is possible to develop sufficiently powerful tools. The notion of semiboundedness specified below is very much inspired by the concepts and requirements of mathematical physics and provides a unifying framework for a substantial class of representations and several interesting phenomena.

The selfadjoint operators  $\overline{d\pi}(x)$  permit us to formulate suitable regularity conditions for unitary representations to specify interesting classes of representations for which a powerful theory can be developed. To ensure that Lie theoretic methods can be applied, we have to require that the representation  $(\pi, \mathcal{H})$  is *smooth*, i.e., that the subspace

$$\mathcal{H}^\infty := \mathcal{H}^\infty(\pi) := \{v \in \mathcal{H} : \pi^v \in C^\infty(G, \mathcal{H})\}$$

is dense in  $\mathcal{H}$ .<sup>2</sup> For a smooth unitary representation  $(\pi, \mathcal{H})$ , we consider its *support functional*

$$s_\pi: \mathfrak{g} \rightarrow \mathbb{R} \cup \{\infty\}, \quad s_\pi(x) := \sup \left( \text{Spec}(i\overline{d\pi}(x)) \right).$$

<sup>1</sup>More general examples of this type arise as groups of compactly supported gauge transformations of  $K$ -principal bundles  $P \rightarrow M$  (cf. Section 6).

<sup>2</sup>If  $G$  is finite dimensional, then L. Gårding's Theorem asserts that this is always satisfied [12], but this is not the case for infinite dimensional Lie groups. The representation of the additive Banach-Lie group  $G := L^2([0, 1], \mathbb{R})$  on  $\mathcal{H} = L^2([0, 1], \mathbb{C})$  by  $\pi(g)f := e^{ig}f$  is continuous with  $\mathcal{H}^\infty = \{0\}$  ([2]).

This is a lower semicontinuous (its epigraph is closed), positively homogeneous convex function on  $\mathfrak{g}$  which is invariant under the adjoint action of  $G$  on  $\mathfrak{g}$  (cf. Section 2). Its natural “domain of regularity” is the open convex cone

$$W_\pi := \{x \in \mathfrak{g} : (\exists U \text{ open}, x \in U) \sup(s_\pi|_U) < \infty\}.$$

We say that  $\pi$  is *semibounded* if  $W_\pi$  is non-empty and that  $\pi$  is *bounded* if  $W_\pi = \mathfrak{g}$  (cf. [33]).

It turns out that, to a large extent, the methods of classical harmonical analysis on locally compact groups can be extended to semibounded unitary representations of infinite dimensional Lie groups. Below we shall discuss some of the cornerstones of this theory:

- The existence of  $C^*$ -algebras  $\mathcal{A}$  whose representations are in one-to-one correspondence with those representations of  $G$  satisfying a certain spectral condition (Theorem 4.3). Here the method of smoothing operators developed recently in [42] is the key to complete the picture (Section 3).
- The Spectral Theorem for semibounded representations of locally convex spaces, resp., abelian Lie groups [33, Thm. 4.1] (Remark 4.4).
- The method of holomorphic induction which is presently the most effective tool to classify semibounded representations (see [36] for the case of Banach–Lie groups and [37, App. C] for generalizations to Fréchet–Lie groups) (Section 7).
- Some classification results for bounded representations that are particularly important as input for holomorphic induction ([29, 38, 20, 4]) (Sections 5, 6).
- Some classification results for semibounded representations of various classes of Lie groups ([35, 37, 40]) (Sections 8, 9, 10).

In addition to their tractability, the restriction to the class of semibounded representations is to a large extent motivated by the fact that representations arising in quantum physics are often characterized by the requirement that the Lie algebra  $\mathfrak{g}$  of  $G$  contains an element  $h$ , corresponding to the Hamiltonian of the underlying physical system, for which the spectrum of the operator  $-i\mathfrak{d}\pi(h)$  is bounded from below. These representations are called *positive energy representations* (for the appearance of such conditions is physics, see [54, 55], [52], [7], [28], [48], [8], [5], [10], [1]). Semiboundedness is a stable version of the positive energy condition. It means that the selfadjoint operators  $i\mathfrak{d}\pi(x)$  from the derived representation are uniformly bounded from below for all  $x$  in some non-empty open subset of  $\mathfrak{g}$ .

## 2. Momentum sets of smooth unitary representations

In this section we introduce the momentum map and the momentum set of a smooth unitary representation of a Lie group  $G$ .

Let  $(\pi, \mathcal{H})$  be a smooth unitary representation of  $G$ , so that the subspace  $\mathcal{H}^\infty$  of smooth vectors is dense. On  $\mathcal{H}^\infty$  the derived representation  $\mathfrak{d}\pi$  of the Lie algebra  $\mathfrak{g} = \mathbf{L}(G)$  is defined by

$$\mathfrak{d}\pi(x)v := \left. \frac{d}{dt} \right|_{t=0} \pi(\exp tx)v.$$

The invariance of  $\mathcal{H}^\infty$  under  $\pi(G)$  implies that, for  $x \in \mathfrak{g}$ , the operator  $i\mathfrak{d}\pi(x)$  on  $\mathcal{H}^\infty$  is essentially selfadjoint (cf. [49, Thm. VIII.10]) and that its closures coincides with the selfadjoint generator  $i\overline{\mathfrak{d}\pi}(x)$  of the unitary one-parameter group  $\pi_x(t) := \pi(\exp tx)$ .

**Definition 2.1.** (a) Let  $\mathbb{P}(\mathcal{H}^\infty) = \{[v] := \mathbb{C}v : 0 \neq v \in \mathcal{H}^\infty\}$  denote the projective space of the subspace  $\mathcal{H}^\infty$  of smooth vectors. The map

$$\Phi_\pi : \mathbb{P}(\mathcal{H}^\infty) \rightarrow \mathfrak{g}' \quad \text{with} \quad \Phi_\pi([v])(x) = \frac{1}{i} \frac{\langle \mathfrak{d}\pi(x)v, v \rangle}{\langle v, v \rangle}$$

is called the *momentum map of the unitary representation*  $\pi$ . The operator  $i\mathfrak{d}\pi(x)$  is symmetric so that the right hand side is real, and since  $v$  is a smooth vector, it defines a continuous linear functional on  $\mathfrak{g}$ . We also observe that we have a natural action of  $G$  on  $\mathbb{P}(\mathcal{H}^\infty)$  by  $g.[v] := [\pi(g)v]$ , and the relation

$$\pi(g)\mathfrak{d}\pi(x)\pi(g)^{-1} = \mathfrak{d}\pi(\text{Ad}(g)x)$$

immediately implies that  $\Phi_\pi$  is equivariant with respect to the coadjoint action of  $G$  on the topological dual space  $\mathfrak{g}'$ .<sup>3</sup>

(b) The weak\*-closed convex hull  $I_\pi \subseteq \mathfrak{g}'$  of the image of  $\Phi_\pi$  is called the (*convex*) *momentum set of*  $\pi$ . In view of the equivariance of  $\Phi_\pi$ , it is an  $\text{Ad}^*(G)$ -invariant closed convex subset of  $\mathfrak{g}'$ .

The momentum set  $I_\pi$  provides complete information on the extreme spectral values of the operators  $i\mathfrak{d}\pi(x)$ :

$$\sup(\text{Spec}(i\mathfrak{d}\pi(x))) = s_\pi(x) = \sup\langle I_\pi, -x \rangle \quad \text{for} \quad x \in \mathfrak{g} \quad (2)$$

(cf. [32, Lemma 5.6]). This relation shows that  $s_\pi$  is the *support functional* of the convex subset  $I_\pi \subseteq \mathfrak{g}'$ , which implies that it is lower semicontinuous and convex. It is obviously positively homogeneous. The semibounded unitary representations are those for which  $s_\pi$ , resp., the momentum set  $I_\pi$ , contains significant information on  $\pi$ . For these the set  $I_\pi$  is *semi-equicontinuous* in the sense that its support function  $s_\pi$  is bounded in a neighborhood of some  $x_0 \in \mathfrak{g}$ .

**Remark 2.2.** (Physical interpretation) (a) In quantum mechanics the space  $\mathbb{P}(\mathcal{H}^\infty)$  is interpreted as the state space of a physical system and the selfadjoint operator

<sup>3</sup>If  $G$  is a Banach–Lie group, then  $\mathcal{H}^\infty$  carries a natural Fréchet structure for which the  $G$ -action on the complex Fréchet–Kähler manifold  $\mathbb{P}(\mathcal{H}^\infty)$ , endowed with the Fubini–Study metric, is Hamiltonian with momentum map  $\Phi_\pi$  (cf. [34, Thm. 4.5]). This motivates the terminology.

$-i\mathfrak{d}\pi(x)$  represents an observable. If  $P$  is the spectral measure of this operator, then  $-i\mathfrak{d}\pi(x) = \int_{\mathbb{R}} t dP(t)$  and

$$\Phi_{\pi}([v])(x) = \frac{1}{\langle v, v \rangle} \int_{\mathbb{R}} t d\langle P(t)v, v \rangle$$

is the expectation value of the observable  $-i\mathfrak{d}\pi(x)$  in the state  $[v]$ .

(b) The groups arising as symmetry groups in quantum physics have natural projective representations (as symmetry groups of the quantum state space) satisfying a suitable positive energy condition (for the observable corresponding to the Hamiltonian, resp. the energy). Accordingly, one often has to pass to central extensions to find unitary representations.

From [33, Thm. 3.1] and its proof we obtain the following characterization of bounded representations:

**Theorem 2.3.** *For a smooth unitary representation  $(\pi, \mathcal{H})$ , the following are equivalent:*

- (i)  $\pi$  is bounded.
- (ii)  $\pi: G \rightarrow \mathbf{U}(\mathcal{H})$  is smooth with respect to the Banach–Lie group structure on  $\mathbf{U}(\mathcal{H})$  defined by the norm topology.
- (iii)  $\mathcal{H}^{\infty} = \mathcal{H}$  and  $\mathfrak{d}\pi: \mathfrak{g} \rightarrow B(\mathcal{H})$  is a continuous linear map.
- (iv)  $I_{\pi}$  is equicontinuous.

### 3. Smoothing operators

A key tool to construct  $C^*$ -algebras for smooth unitary representations is the concept of a smoothing operator introduced in [42].

**Definition 3.1.** For a smooth unitary representation  $(\pi, \mathcal{H})$  we call a bounded operator  $A \in B(\mathcal{H})$  *smoothing* if  $A\mathcal{H} \subseteq \mathcal{H}^{\infty}$ .<sup>4</sup>

**Remark 3.2.** (a)  $\mathbf{1} = \text{id}_{\mathcal{H}}$  is smoothing if and only if  $\pi$  is bounded (cf. Theorem 3.3 below).

(b) If  $G$  is finite dimensional and  $f \in C_c^{\infty}(G)$ , then  $\pi(f) = \int_G f(g)\pi(g) dg$  is smoothing ([12]).

Other types of smoothing operators can be obtained from a basis  $x_1, \dots, x_n$  of  $\mathfrak{g}$ . *Nelson's Laplacian*  $\Delta := \sum_{j=1}^n \mathfrak{d}\pi(x_j)^2$  is selfadjoint with non-positive spectrum,

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<sup>4</sup> In [22]  $A \in B(\mathcal{H})$  is called a *Schwartz operator* if all operators  $\mathfrak{d}\pi(D_1)A\mathfrak{d}\pi(D_2)$ ,  $D_1, D_2 \in U(\mathfrak{g})$  are bounded. In view of Theorem 3.3, for every Schwartz operator  $A \in B(\mathcal{H})$  both  $A$  and  $A^*$  are smoothing. It is an interesting open problem whether the converse is also true. In [22] Schwartz operators are studied for the Schrödinger representation of the Heisenberg group on  $\mathbb{R}^{2N}$ .

so that the operators  $(e^{t\Delta})_{t>0}$  (the corresponding “heat semigroup”) define a one-parameter group of selfadjoint contractions. It follows from [44, Cor. 9.3] that these are smoothing operators.

(c) If  $\beta: H \rightarrow G$  is a smooth homomorphism of Lie groups and  $H$  is finite dimensional with  $\mathcal{H}^\infty = \mathcal{H}^\infty(\pi \circ \beta)$ , then, for every  $f \in C_c^\infty(H)$ ,  $\pi(f) = \int_H f(h)\pi(\beta(h)) dh$  is smoothing by (b).

The following theorem is the main result on smoothing operators in [42]. Its main power lies in the fact that the rather weak smoothing condition implies the smoothness of the multiplication maps in the operator norm.

**Theorem 3.3.** (Characterization Theorem for smoothing operators) *Let  $(\pi, \mathcal{H})$  be a smooth unitary representation of a metrizable Lie group  $G$  with exponential function. Then  $A \in B(\mathcal{H})$  is smoothing if and only if  $G \rightarrow B(\mathcal{H}), g \mapsto \pi(g)A$  is smooth with respect to the norm topology on  $B(\mathcal{H})$ . If, in addition,  $G$  is Fréchet, then this is also equivalent to*

- (i)  $A\mathcal{H} \subseteq \mathcal{D}(\overline{d\pi}(x_1) \cdots \overline{d\pi}(x_n))$  for  $x_1, \dots, x_n \in \mathfrak{g}, n \in \mathbb{N}$ .
- (ii) All operators  $A^* \overline{d\pi}(x_1) \cdots \overline{d\pi}(x_n)$ ,  $x_1, \dots, x_n \in \mathfrak{g}, n \in \mathbb{N}$ , are bounded.

The following theorem is an important source of smoothing operators ([42, Thm. 3.4]):

**Theorem 3.4** (Zellner’s Smooth Vector Theorem). *If  $(\pi, \mathcal{H})$  is semibounded and  $x_0 \in W_\pi$ , then  $\mathcal{H}^\infty = \mathcal{H}^\infty(\pi_{x_0})$  (cf. (1)). In particular, the operators*

$$e^{i\overline{d\pi}(x_0)} \quad \text{and} \quad \int_{\mathbb{R}} f(t)\pi(\exp tx_0) dt = \widehat{f}(i\overline{d\pi}(x_0)), \quad f \in \mathcal{S}(\mathbb{R}),$$

*are smoothing.*

The essentially selfadjoint operator  $i\overline{d\pi}(x_0)$  plays for a semibounded representation a similar role as Nelson’s Laplacian  $\Delta$  for a representation of a finite dimensional Lie group. This clearly demonstrates that, although one has very general tools that work for all representations of finite dimensional Lie groups (such as Nelson’s heat semigroup and smoothing by convolution), specific classes of representations of infinite dimensional groups (such as semibounded ones) require specific but nevertheless equally powerful methods (such as Zellner’s smoothing operators).

## 4. $C^*$ -algebras

In the unitary representation theory of a finite dimensional Lie group  $G$ , a central tool is the convolution algebra  $L^1(G)$ , resp., its enveloping  $C^*$ -algebra  $C^*(G)$ , whose construction is based on the Haar measure, whose existence follows from the local compactness of  $G$ . Since the non-degenerate representations of  $C^*(G)$  are in one-to-one correspondence with continuous unitary representations of  $G$ ,

the full power of the rich theory of  $C^*$ -algebras can be used to study unitary representations of  $G$  ([9]).

For infinite dimensional Lie groups there is no immediate analog of the convolution algebra  $L^1(G)$ , so that we cannot hope to find a  $C^*$ -algebra whose representations are in one-to-one correspondence with all unitary representations of  $G$ . However, in [19] H. Grundling introduced the notion of a *host algebra* of a topological group  $G$  and we shall see below that this concept provides natural  $C^*$ -algebras whose representations are in one-to-one correspondence with certain semibounded representations of  $G$ .

**Definition 4.1.** Let  $\mathcal{A}$  be a  $C^*$ -algebra,  $M(\mathcal{A})$  its multiplier algebra,<sup>5</sup> and let  $G$  be a topological group. We consider a homomorphism  $\eta: G \rightarrow U(M(\mathcal{A}))$  into the unitary group of the  $C^*$ -algebra  $M(\mathcal{A})$ . Then the pair  $(\mathcal{A}, \eta)$  is called a *host algebra for  $G$*  if, for each non-degenerate representation  $\rho$  of  $\mathcal{A}$  and its canonical extension  $\tilde{\rho}$  to  $M(\mathcal{A})$ , the unitary representation  $\rho_G := \tilde{\rho} \circ \eta$  of  $G$  is continuous and determines  $\rho$  uniquely.

In this sense,  $\mathcal{A}$  is hosting a certain class of representations of  $G$ . We identify the set  $\hat{\mathcal{A}}$  of equivalence classes of irreducible representations of  $\mathcal{A}$  via  $[\rho] \mapsto [\rho_G]$  with a subset  $\hat{G}_{\mathcal{A}}$  of the *unitary dual*  $\hat{G}$  of  $G$  (the set of equivalence classes of irreducible unitary representations of  $G$ ). A host algebra  $\mathcal{A}$  is called *full* if it is hosting all continuous unitary representations of  $G$ .

If  $G$  is locally compact, then  $\mathcal{A} = C^*(G)$ , with  $\eta$  specified by

$$\eta(g)f = \delta_g * f \quad \text{for } f \in L^1(G), g \in G,$$

defines a full host algebra [9]. But to which extent do infinite dimensional Lie groups possess host algebras? If  $G = (E, +)$  is an infinite dimensional locally convex space, then the set of equivalence classes of irreducible unitary representations can be identified with the dual space  $E'$ , and since this space carries no natural locally compact topology, one cannot expect the existence of a full host algebra in general. Therefore one is looking for host algebras that accommodate certain classes of unitary representations. For bounded representations this is easy:

**Example 4.2.** Let  $(\pi, \mathcal{H})$  be a bounded unitary representation of the Lie group  $G$  and consider the  $C^*$ -subalgebra  $\mathcal{A}_{\pi} := C^*(\pi(G)) \subseteq B(\mathcal{H})$  generated by  $\pi(G)$ . Then  $\mathcal{A}_{\pi}$  is unital, so that  $\mathcal{A}_{\pi} \cong M(\mathcal{A}_{\pi})$  and  $\eta := \pi: G \rightarrow U(\mathcal{A}_{\pi})$  is smooth with respect to the norm topology on the Banach–Lie group  $U(\mathcal{A}_{\pi})$ . This implies that, for every representation  $(\rho, \mathcal{K})$  of  $\mathcal{A}_{\pi}$ , the representation  $\rho_G := \rho \circ \pi$  is bounded and, since  $\text{span } \pi(G)$  is dense in  $\mathcal{A}_{\pi}$ , it determines  $\rho$  uniquely. Therefore  $(\mathcal{A}_{\pi}, \pi)$  is a host algebra of  $G$  whose representations correspond to certain bounded representations of  $G$ .

From this observation and Theorem 2.3 one easily obtains for every continuous seminorm  $p$  on  $\mathfrak{g}$  a unital host algebra  $(\mathcal{A}_p, \eta_p)$  whose representations are precisely

<sup>5</sup>If  $\mathcal{A}$  is realized as a closed subalgebra of some  $B(\mathcal{H})$ , then

$$M(\mathcal{A}) \cong \{B \in B(\mathcal{H}) : (\forall A \in \mathcal{A}) \ BA, AB \in \mathcal{A}\}.$$

those smooth representations  $(\pi, \mathcal{H})$  of  $G$  satisfying  $\|\mathrm{d}\pi(x)\| \leq p(x)$  for  $x \in \mathfrak{g}$  (cf. [30, §III.2]), and this condition is equivalent to

$$I_\pi \subseteq \{\lambda \in \mathfrak{g}' : (\forall x \in \mathfrak{g}) |\lambda(x)| \leq p(x)\},$$

which is an equicontinuous subset of  $\mathfrak{g}'$ .

It turns out that Theorem 3.4 is precisely what is needed to generalize the preceding construction to semibounded representations. Here one has to deal with non-unital algebras and smoothing operators of the form  $e^{i\overline{\mathrm{d}\pi}(x_0)}$  that lead for every semibounded representation  $(\pi, \mathcal{H})$  and  $x_0 \in W_\pi$  to the host algebra  $\mathcal{A} := C^*(\pi(G)e^{i\overline{\mathrm{d}\pi}(x_0)}\pi(G))$ . Putting everything together, we obtain:

**Theorem 4.3.** ([42, Cor. 4.9]) *Let  $C \subseteq \mathfrak{g}'$  be a weak\*-closed  $\mathrm{Ad}^*(G)$ -invariant subset which is semi-equicontinuous in the sense that its support function  $s_C(x) := \sup\langle C, x \rangle$  is bounded in a neighborhood of some  $x_0 \in \mathfrak{g}$ . Then there exists for every semibounded representation  $(\pi, \mathcal{H})$  a host algebra  $(\mathcal{A}_C, \eta_C)$  of  $G$  whose representations correspond to those semibounded unitary representations  $(\pi, \mathcal{H})$  of  $G$  for which  $s_\pi \leq s_C$ , resp.,  $I_\pi \subseteq -C$ .*

**Remark 4.4.** ([32, Prop. 6.13]) (a) If  $\pi$  is (semi)bounded, then  $I_\pi$  is a (locally) compact subset of  $\mathfrak{g}'$ , endowed with the weak\*-topology and for every  $x_0 \in W_\pi$  the map  $I_\pi \rightarrow \mathbb{R}, \alpha \mapsto \alpha(x_0)$  is proper.

(b) If  $G = (E, +)$  is the additive group of a locally convex space, then we identify the topological dual space  $E'$  with the character group  $\widehat{G}$  by  $\chi_\alpha(v) := e^{i\alpha(v)}$ . Then there exists for every semibounded representation  $(\pi, \mathcal{H})$  a spectral measure  $P$  on the locally compact space  $I_\pi \subseteq E'$  with  $\pi(v) = \int_{E'} e^{i\alpha(v)} dP(\alpha)$  and the  $C^*$ -algebra from Theorem 4.3 is isomorphic to  $C_0(\mathrm{supp}(P))$ , where  $\mathrm{supp}(P) \subseteq I_\pi$  is the support of  $P$  (cf. [33, Thm. 4.1]).

(c) If  $C_1 \subseteq C_2$  are  $\mathrm{Ad}^*(G)$ -invariant weak\*-closed convex equicontinuous subsets, then the construction of the host algebras  $\mathcal{A}_{C_1}$  and  $\mathcal{A}_{C_2}$  provides a morphism  $\mathcal{A}_{C_2} \twoheadrightarrow \mathcal{A}_{C_1}$ . Since the collection of all  $\mathrm{Ad}^*(G)$ -invariant weak\*-closed convex equicontinuous subsets is directed with respect to inclusion, we thus obtain a projective system of  $C^*$ -algebras  $\mathcal{A}_C$  whose representation theory completely describes the semibounded representations of  $G$ . Similar results for finite dimensional groups obtained by holomorphic extensions to complex semigroups can already be found in [30, §XI.6].

**Remark 4.5.** In general, continuous unitary representations of locally convex spaces can not be represented in terms of spectral measures on  $E'$ . This is closely related to the problem of writing the continuous positive definite functions  $\pi^{v,v}(x) := \langle \pi(x)v, v \rangle$  as a Fourier transform

$$\widehat{\mu}(x) = \int_{E'} e^{i\alpha(x)} d\mu(\alpha)$$

of some finite measure  $\mu$  on  $E'$ . If  $E$  is nuclear, then the Bochner–Minlos Theorem [13] ensures the existence of such measures and hence also of spectral measures representing unitary representations. However, if  $E$  is an infinite dimensional Hilbert



space  $E$ , then the continuous positive definite function  $\varphi(v) := e^{-\|v\|^2/2}$  is not the Fourier transform of a positive measure on  $E'$  ([56, Ex. 17.1]). Note that  $E$  is not nuclear, so that the Bochner–Minlos Theorem does not apply. Therefore it is quite remarkable that nuclearity assumptions are not needed to deal with semibounded representations because the domains of the spectral measures are locally compact. It is easy to show that, if  $E$  is separable and metrizable, then a spectral measure for a unitary representation  $(\pi, \mathcal{H})$  exists if and only if  $\pi$  is a direct sum of bounded representations.

The main motivation to find host algebras is that they provide very well developed tools to decompose the representations into irreducible ones: If  $(\mathcal{A}, \eta)$  is a separable host algebra of  $G$  which is of type I,<sup>6</sup> then the following abstract disintegration theorem applies immediately to all  $\mathcal{A}$ -representations  $\rho_G$  of  $G$  and thus reduces the classification problems for semibounded representations to the classification of the irreducible representations and the spectral multiplicity theory of measures on  $\widehat{G}_{\mathcal{A}}$ :

**Theorem 4.6** (Abstract Disintegration Theorem). ([9, Th. 8.6.6]) *Let  $\mathcal{A}$  be a separable type I  $C^*$ -algebra.*

- (i) *Every Borel measure  $\mu$  on  $\widehat{\mathcal{A}}$  defines a multiplicity free direct integral representation  $\pi_{\mu} := \int_{\widehat{\mathcal{A}}}^{\oplus} \pi d\mu(\pi)$ . Two such representations  $\pi_{\mu}$  and  $\pi_{\nu}$  are equivalent if and only the measure classes of  $\mu$  and  $\nu$  coincide.*
- (iii) *For every separable representation  $(\pi, \mathcal{H})$ , there exist mutually disjoint measures  $(\mu_n)_{n \in \mathbb{N} \cup \{\infty\}}$  such that*

$$\pi \cong \pi_{\mu_1} \oplus 2 \cdot \pi_{\mu_2} \oplus \cdots \oplus \aleph_0 \cdot \pi_{\mu_{\infty}}.^7$$

*The measure classes of  $(\mu_n)_{n \in \mathbb{N} \cup \{\infty\}}$  are uniquely determined by  $\pi$ .*

## 5. Representations of unitary groups

Let  $\mathcal{H}$  be an infinite dimensional complex Hilbert space. We write  $U(\mathcal{H})_n$  for the Banach–Lie group obtained by endowing the unitary group  $U(\mathcal{H})$  with the norm topology and  $U(\mathcal{H})_s$  for the topological group structure obtained from the strong operator topology, i.e., the topology of pointwise convergence.

The representation theory of infinite dimensional unitary groups began with I. E. Segal’s paper [53], where he studies so-called *physical representations* of the full group  $U(\mathcal{H})_s$ . These are characterized by the condition that their differential maps finite rank hermitian projections to positive operators. Segal shows that physical representations are direct sums of irreducible ones, which are precisely

<sup>6</sup>This means that, for every (non-zero) irreducible representations  $(\rho, \mathcal{K})$  of  $\mathcal{A}$ , the image  $\rho(\mathcal{A})$  contains a non-zero compact operator ([50]), and this in turn implies that  $K(\mathcal{K}) \subseteq \rho(\mathcal{A})$ .

<sup>7</sup>For a representation  $(\rho, \mathcal{K})$  and a cardinal  $n$ , we write  $n \cdot \rho$  for the representation  $\rho \otimes \mathbf{1}$  on  $\mathcal{K} \widehat{\otimes} \ell^2(\mathbf{n})$ , where  $\mathbf{n}$  is a set of cardinality  $n$ .

those occurring in the decomposition of finite tensor products  $\mathcal{H}^{\otimes N}$ ,  $N \in \mathbb{N}_0$ . This tensor product decomposes as in classical Schur–Weyl theory:

$$\mathcal{H}^{\otimes N} \cong \bigoplus_{\lambda \in \text{Part}(N)} \mathbb{S}_\lambda(\mathcal{H}) \otimes \mathcal{M}_\lambda, \quad (3)$$

where  $\text{Part}(N)$  is the set of all partitions  $\lambda = (\lambda_1, \dots, \lambda_n)$  of  $N$ , and  $\mathbb{S}_\lambda(\mathcal{H})$  is an irreducible unitary representation of  $U(\mathcal{H})$  (called a *Schur representation*), and  $\mathcal{M}_\lambda$  is the corresponding irreducible representation of the symmetric group  $S_N$ , hence in particular finite dimensional.<sup>8</sup> In particular,  $\mathcal{H}^{\otimes N}$  is a finite sum of irreducible representations of  $U(\mathcal{H})$ .

The representation theory of the Banach–Lie group

$$U_\infty(\mathcal{H}) = U(\mathcal{H}) \cap (1 + K(\mathcal{H})),$$

where  $\mathcal{H}$  is a separable complex Hilbert space and  $K(\mathcal{H})$  the space of compact operators, was developed by Kirillov and Olshanski in [23] and [45, Thm. 1.11]. They show that all continuous representations of  $U_\infty(\mathcal{H})$  are direct sums of irreducible representations and that, for  $\mathbb{K} = \mathbb{C}$ , all irreducible representations are of the form  $\mathbb{S}_\lambda(\mathcal{H}) \otimes \mathbb{S}_\mu(\overline{\mathcal{H}})$ , where  $\overline{\mathcal{H}}$  is the space  $\mathcal{H}$ , endowed with the opposite complex structure. They also obtained generalizations for the corresponding groups over real and quaternionic Hilbert spaces. It follows in particular that all irreducible representations  $(\pi, \mathcal{H}_\pi)$  of the Banach–Lie group  $U_\infty(\mathcal{H})$  are bounded. The classification of the bounded unitary representations of the Banach–Lie group  $U_p(\mathcal{H}) := U(\mathcal{H}) \cap (1 + B_p(\mathcal{H}))$  (where  $B_p(\mathcal{H})$  is the  $p$ th Schatten ideal) does not depend on  $p$  as long as  $1 < p \leq \infty$ , but, for  $p = 1$ , factor representations of type II and III exist (see [6] for  $p = 2$ , and [29] for the general case). Dropping the boundedness assumptions leads to non-type I representations of  $U_p(\mathcal{H})$ ,  $p < \infty$  (cf. [6, Thm. 5.5]).

These results clearly show that the group  $U_\infty(\mathcal{H})$  is singled out among all its relatives  $U_p(\mathcal{H})$  by the fact that its unitary representation theory is well-behaved. If  $\mathcal{H}$  is separable, then  $U_\infty(\mathcal{H})$  is separable, so that its cyclic representations are separable as well. Hence there is no need to discuss inseparable representations for this group. This is different for the Banach–Lie group  $U(\mathcal{H})_n$  which has many inseparable bounded irreducible unitary representations coming from irreducible representations of the Calkin algebra  $B(\mathcal{H})/K(\mathcal{H})$ . The following theorem was an amazing achievement of D. Pickrell [46], showing that restricting attention to representations on separable spaces tames the representation theory of  $U(\mathcal{H})_n$  in the sense that all its separable representations are actually continuous with respect to the strong operator topology, i.e., continuous representations of  $U(\mathcal{H})_s$ .

**Theorem 5.1** (Pickrell’s Continuity Theorem). *Every separable unitary representation<sup>9</sup> of  $U(\mathcal{H})_n$  is also continuous for the strong operator topology, hence a representation of  $U(\mathcal{H})_s$ .*

<sup>8</sup>We refer to [3] for an extension of Schur–Weyl theory to irreducible representations of  $C^*$ -algebras.

<sup>9</sup>Recall that we included continuity in the definition of a unitary representation.

Building on the fact that the identity component  $U_\infty(\mathcal{H})_0$  is dense in  $U(\mathcal{H})_s$  and extending the Kirillov–Olshanski classification to non-separable Hilbert spaces, we show in [38] that the representations of  $U_\infty(\mathcal{H})_0$  and  $U(\mathcal{H})_s$  coincide, more precisely:

**Theorem 5.2.** *Let  $\mathcal{H}$  be an infinite dimensional real, complex or quaternionic Hilbert space. Then every continuous unitary representation of  $U_\infty(\mathcal{H})_0$  extends uniquely to a continuous representation of  $U(\mathcal{H})_s$ . All unitary representations of these groups are direct sums of irreducible ones which are of the form  $\mathbb{S}_\lambda(\mathcal{H}) \otimes \mathbb{S}_\mu(\overline{\mathcal{H}})$ , hence in particular bounded representations of  $U(\mathcal{H})_n$ .*

In view of the preceding results, the separable representation theory of the Lie group  $U(\mathcal{H})_n$  very much resembles the representation theory of a compact group, and so does the representation theory of  $U_\infty(\mathcal{H})$ .

## 6. Bounded representations of gauge groups

Let  $M$  be a smooth  $\sigma$ -compact manifold,  $K$  a compact connected Lie group and  $q: P \rightarrow M$  a  $K$ -principal bundle. We consider the Lie group  $G := \text{Gau}_c(P)$  of compactly supported gauge transformations and observe that it can be realized as

$$\text{Gau}_c(P) \cong C_G^\infty(P, K) := \{f \in C_c^\infty(P, K) : (\forall p \in P)(\forall k \in K) f(p.k) = k^{-1}f(p)k\}.$$

Its Lie algebra  $\mathfrak{g}$  is

$$\mathfrak{gau}_c(P) \cong C_G^\infty(P, \mathfrak{k}) := \{\xi \in C_c^\infty(P, \mathfrak{k}) : (\forall p \in P)(\forall k \in K) \xi(p.k) = \text{Ad}(k)^{-1}\xi(p)\}.$$

Bounded representations of  $G$  are easy to construct by evaluations. To see this, let  $x \in P$ ,  $(\rho, V_\rho)$  be an irreducible representation of  $K$  (which is automatically finite dimensional). Then

$$\pi_{x,\rho}(f) := \rho(f(x)) \quad \text{for } f \in G, x \in P$$

defines a finite dimensional irreducible representation of  $G$ . Clearly  $\pi_{x,\rho} \cong \pi_{y,\rho}$  if  $q(x) = q(y)$ . For finite subsets  $\mathbf{x} = \{x_1, \dots, x_N\} \subseteq P$  for which the points  $q(x_i) \in M$  are pairwise different, the representation

$$\pi_{\mathbf{x},\rho} := \bigotimes_{x \in \mathbf{x}} \pi_{x,\rho_x}$$

of  $G$  is also bounded and irreducible [20].

We can even go further: Let  $\mathbf{x} \subseteq P$  be a subset mapped injectively by  $q$  to a locally finite subset of  $M$ . Then we assign to every collection  $\rho = (\rho_x, V_x)_{x \in \mathbf{x}}$  of irreducible representations of  $K$  the UHF- $C^*$ -algebra

$$\mathcal{A}_{\mathbf{x},\rho} := \bigotimes_{x \in \mathbf{x}} B(V_x). \quad \text{Then } \eta_{\mathbf{x},\rho} := \bigotimes_{x \in \mathbf{x}} \pi_{x,\rho_x} : C_c^\infty(M, K) \rightarrow U(\mathcal{A}_{\mathbf{x},\rho})$$

defines a smooth homomorphism into the unitary group of the  $C^*$ -algebra  $\mathcal{A}_{\mathbf{x},\rho}$  and it is not hard to see that the image of  $\eta_{\mathbf{x},\rho}$  generates  $\mathcal{A}_{\mathbf{x},\rho}$ , so that we obtain a host algebra  $(\mathcal{A}_{\mathbf{x},\rho}, \eta_{\mathbf{x},\rho})$  of  $G$  (cf. Example 4.2). The following theorem reduces the classification of the bounded representations of  $G$  completely to questions on  $C^*$ -algebras (cf. [20]).

**Theorem 6.1.** *Every bounded irreducible representation  $\pi$  of the identity component  $\text{Gau}_c(P)_0$  is of the form  $\beta \circ \eta_{\mathbf{x},\rho}$  for some irreducible representation  $\beta$  of  $\mathcal{A}_{\mathbf{x},\rho}$ . If  $M$  is compact, then  $\mathbf{x}$  is finite and  $\pi \cong \pi_{\mathbf{x},\rho}$ .*

**Remark 6.2.** (a) If the bundle  $P$  is trivial, then  $G \cong C_c^\infty(M, K)$  is a mapping group. It is contained in the Banach–Lie group  $C_0(M, K)_0$  of maps vanishing at infinity. In [20] it is also shown that every irreducible bounded representation of  $G$  that extends to a bounded representation of this group is of the form  $\pi_{\mathbf{x},\rho}$  for a finite subset  $\mathbf{x}$ .

(b) A description of the irreducible representations of the UHF- $C^*$ -algebras  $\mathcal{A} = \bigotimes_{n \in \mathbb{N}} M_{d_n}(\mathbb{C})$  can be obtained from the work of Glimm [14] and Powers [47]. One of their main results is that all irreducible representations of  $\mathcal{A}$  are twists of infinite tensor products of irreducible representations  $\bigotimes_x (V_x, v_x)$  ( $v_x \in V_x$  a unit vector) by an automorphism of  $\mathcal{A}$ . Since  $\mathcal{A}$  has a large automorphism group, this implies in particular that not all bounded irreducible representations  $G$  are tensor products of irreducible representations of the factors.

## 7. From bounded to semibounded representations by holomorphic induction

We now describe a complex geometric technique to classify semibounded representations. For the results in this section we refer to [37, App. C] for the Fréchet case and to [36] for a more detailed exposition of these methods for Banach–Lie groups.

Let  $G$  be a connected Fréchet–BCH–Lie group with Lie algebra  $\mathfrak{g}$ .<sup>10</sup> We further assume that there exists a complex BCH–Lie group  $G_{\mathbb{C}}$  with Lie algebra  $\mathfrak{g}_{\mathbb{C}}$  and a natural map  $\eta: G \rightarrow G_{\mathbb{C}}$  for which  $\mathbf{L}(\eta)$  is the inclusion  $\mathfrak{g} \hookrightarrow \mathfrak{g}_{\mathbb{C}}$ . Let  $H \subseteq G$  be a Lie subgroup with Lie algebra  $\mathfrak{h}$  for which  $M := G/H$  carries the structure of a smooth manifold with a smooth  $G$ -action. We also assume the existence of closed  $\text{Ad}(H)$ -invariant subalgebras  $\mathfrak{p}^{\pm} \subseteq \mathfrak{g}_{\mathbb{C}}$  with  $\overline{\mathfrak{p}^{\pm}} = \mathfrak{p}^{\mp}$  for  $x + iy := x - iy$ ,  $x, y \in \mathfrak{g}$ , and for which we have a topological direct sum decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{p}^-. \quad (\text{SC})$$

We put

$$\mathfrak{q} := \mathfrak{p}^+ \rtimes \mathfrak{h}_{\mathbb{C}} \quad \text{and} \quad \mathfrak{p} := \mathfrak{g} \cap (\mathfrak{p}^+ \oplus \mathfrak{p}^-),$$

<sup>10</sup>A Lie group  $G$  is called *locally exponential* if  $\exp: \mathfrak{g} \rightarrow G$  maps some 0-neighborhood in  $\mathfrak{g}$  diffeomorphically to an open 1-neighborhood in  $G$ . If, in addition,  $G$  is an *analytic Lie group* and the local diffeomorphism is bianalytic, then  $G$  is called a *BCH–Lie group* because this implies that the Baker–Campbell–Hausdorff (BCH) series defines a local analytic group structure on some 0-neighborhood in  $\mathfrak{g}$  (cf. [31, §IV.1]).

so that  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$  is a topological direct sum. We assume that there exist open symmetric convex 0-neighborhoods  $U_{\mathfrak{g}_{\mathbb{C}}} \subseteq \mathfrak{g}_{\mathbb{C}}$ ,  $U_{\mathfrak{p}} \subseteq \mathfrak{p} \cap U_{\mathfrak{g}_{\mathbb{C}}}$ ,  $U_{\mathfrak{h}} \subseteq \mathfrak{h} \cap U_{\mathfrak{g}_{\mathbb{C}}}$ ,  $U_{\mathfrak{p}^{\pm}} \subseteq \mathfrak{p}^{\pm} \cap U_{\mathfrak{g}_{\mathbb{C}}}$  and  $U_{\mathfrak{q}} \subseteq \mathfrak{q} \cap U_{\mathfrak{g}_{\mathbb{C}}}$  such that the BCH-product  $x * y = x + y + \frac{1}{2}[x, y] + \dots$  is defined and holomorphic on  $U_{\mathfrak{g}_{\mathbb{C}}} \times U_{\mathfrak{g}_{\mathbb{C}}}$ , and the following maps are analytic diffeomorphisms onto an open subset:

$$(A1) \quad U_{\mathfrak{p}} \times U_{\mathfrak{h}} \rightarrow \mathfrak{g}, (x, y) \mapsto x * y.$$

$$(A2) \quad U_{\mathfrak{p}} \times U_{\mathfrak{q}} \rightarrow \mathfrak{g}_{\mathbb{C}}, (x, y) \mapsto x * y.$$

$$(A3) \quad U_{\mathfrak{p}^-} \times U_{\mathfrak{q}} \rightarrow \mathfrak{g}_{\mathbb{C}}, (x, y) \mapsto x * y.$$

Then (A1) implies the existence of a smooth manifold structure on  $M = G/H$  on which  $G$  acts analytically. Condition (A2) implies the existence of a complex manifold structure on  $M$  which is  $G$ -invariant and for which  $T_{1H}(M) \cong \mathfrak{g}_{\mathbb{C}}/\mathfrak{q}$ . Finally, (A3) makes the proof of [36, Thm. 2.6] work, so that we can associate to every bounded unitary representation  $(\rho, V)$  of  $H$  a holomorphic Hilbert bundle  $\mathbb{V} := G \times_H V$  over the complex  $G$ -manifold  $M$ .

**Definition 7.1.** We write  $\Gamma(\mathbb{V})$  for the space of holomorphic sections of the holomorphic Hilbert bundle  $\mathbb{V} \rightarrow M = G/H$  on which the group  $G$  acts by holomorphic bundle automorphisms. A unitary representation  $(\pi, \mathcal{H})$  of  $G$  is said to be *holomorphically induced from*  $(\rho, V)$  if there exists a  $G$ -equivariant linear injection  $\Psi: \mathcal{H} \rightarrow \Gamma(\mathbb{V})$  such that the adjoint of the evaluation map  $\text{ev}_{1H}: \mathcal{H} \rightarrow V = \mathbb{V}_{1H}$  defines an isometric embedding  $\text{ev}_{1H}^*: V \hookrightarrow \mathcal{H}$ . If a unitary representation  $(\pi, \mathcal{H})$  holomorphically induced from  $(\rho, V)$  exists, then it is uniquely determined ([36, Def. 3.10]) and we call  $(\rho, V)$  *(holomorphically) inducible*.

This concept of inducibility involves a choice of sign. Replacing  $\mathfrak{p}^+$  by  $\mathfrak{p}^-$  changes the complex structure on  $G/H$  and thus leads to a different class of holomorphically inducible representations of  $H$ .

**Theorem 7.2.** *Suppose that  $(\pi, \mathcal{H})$  is a unitary representation of  $G$  and  $V \subseteq \mathcal{H}$  is an  $H$ -invariant closed subspace such that*

(HI1) *the representation  $(\rho, V)$  of  $H$  on  $V$  is bounded,*

(HI2)  *$V \cap (\mathcal{H}^{\infty})^{\mathfrak{p}^-}$  is dense in  $V$ , and*

(HI3)  *$\pi(G)V$  spans a dense subspace of  $\mathcal{H}$ .*

*Then  $(\pi, \mathcal{H})$  is holomorphically induced from  $(\rho, V)$  and  $\pi(G)' \rightarrow \rho(H)'$ ,  $A \mapsto A|_V$  is an isomorphism of the commutants.*

The preceding theorem implies in particular that  $(\rho, V)$  is irreducible if and only if  $(\pi, \mathcal{H})$  is. All the concrete classification results for semibounded irreducible representations rest on the fact that they can be obtained by bounded representations of suitably chosen subgroups  $H$  for which a classification of the irreducible bounded representations is available.

**Example 7.3.** It is instructive to see how the general method of holomorphic induction matches the classification of irreducible unitary representations of a compact connected Lie group  $G$ . In this case we choose a maximal torus  $H \subseteq G$  and obtain a triangular decomposition as in (SC), where  $\mathfrak{b} = \mathfrak{p}^- \rtimes \mathfrak{h}_{\mathbb{C}}$  is a Borel subalgebra of the complex reductive Lie algebra  $\mathfrak{g}_{\mathbb{C}}$ . Then the bounded irreducible representations of  $H$  are one-dimensional, hence given by characters  $\chi: H \rightarrow \mathbb{T}$ . Such a character is holomorphically inducible if and only if the weight  $d\chi: \mathfrak{h}_{\mathbb{C}} \rightarrow \mathbb{C}$  is  $\mathfrak{b}$ -dominant. We thus obtain the well-known classification of the finite dimensional irreducible representations of  $G$  by  $\mathfrak{b}$ -dominant weights on  $\mathfrak{h}_{\mathbb{C}}$ .

## 8. Hermitian groups

We now explain the main points of the classification of irreducible semibounded representations of hermitian Lie groups carried out in [35].

**Definition 8.1.** (a) A *hermitian Lie group* is a triple  $(G, \theta, \mathbf{d})$ , where  $G$  is a connected Lie group,  $\theta$  an involutive automorphism of  $G$  with the corresponding eigenspace decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ ,  $\mathbf{d} \in \mathfrak{z}(\mathfrak{k})$  (the center of  $\mathfrak{k}$ ) an element for which  $\text{ad } \mathbf{d}|_{\mathfrak{p}}$  is a complex structure, and  $\mathfrak{p}$  carries an  $e^{\text{ad } \mathfrak{k}}$ -invariant Hilbert space structure. We then write  $K := (G^{\theta})_0$  for the identity component of the group of  $\theta$ -fixed points in  $G$  and observe that our assumptions imply that  $G/K$  is a hermitian symmetric space (modeled on a complex Hilbert space).

(b) We call  $(G, \theta, \mathbf{d})$  *irreducible* if the unitary  $K$ -representation on  $\mathfrak{p}$  is irreducible. We say that  $\mathfrak{g}$  is *full* if  $\text{ad } \mathfrak{k} \subseteq \mathfrak{u}(\mathfrak{p}^+)$  is the full derivation algebra with respect to the Jordan product  $[x, y, z] := [[x, \overline{y}], z]$  on the  $i$ -eigenspace  $\mathfrak{p}^+ \subseteq \mathfrak{g}_{\mathbb{C}}$ .

If  $\mathfrak{g}$  is full, then the Lie algebra  $\mathfrak{k}/\mathfrak{z}(\mathfrak{k})$  contains no non-trivial open invariant cones ([35, Lemma 5.10]). If, in addition,  $(G, \theta, \mathbf{d})$  is irreducible and  $(\pi, \mathcal{H})$  an irreducible semibounded representation, then either  $\mathbf{d} \in W_{\pi}$ , i.e.,  $\pi$  is a *positive energy representation*, or the dual representation  $\pi^*$  has this property ([35, Thm. 6.2]). In the former case, it is holomorphically induced from a bounded representation  $(\rho, V)$  of  $K$  ([35, Thm. 6.4]). Therefore the classification of irreducible semibounded representations completely reduces the determination of the irreducible bounded representations of  $K$  which are holomorphically inducible.

**Example 8.2.** If  $G = \mathfrak{p} \rtimes_{\alpha} K$  is a Cartan motion group of the complex Hilbert space  $\mathfrak{p}$ , i.e.,  $[\mathfrak{p}, \mathfrak{p}] = \{0\}$ , then all semibounded representations  $(\pi, \mathcal{H})$  of  $G$  are trivial on  $\mathfrak{p}$  by [35, Thm. 7.1].

However, the Lie algebra has a unique central extension  $\widehat{\mathfrak{g}} = \mathbb{R} \oplus_{\omega} \mathfrak{g}$ , given by the cocycle  $\omega(x, y) := \text{Im} \langle x_{\mathfrak{p}}, y_{\mathfrak{p}} \rangle$  for  $x = x_{\mathfrak{k}} + x_{\mathfrak{p}}$ . If  $\mathfrak{g}$  is full, then  $\widehat{K} \cong \mathbb{R} \times \text{U}(\mathfrak{p})$  and  $(\rho, V)$  is holomorphically inducible if and only if  $-i d\rho(1, 0) \geq 0$  ([35, Thm. 7.2]).

**Example 8.3.** (a) For the case where  $(G, \theta, \mathbf{d})$  is irreducible and full and  $\mathcal{D} := G/K$  is an infinite dimensional symmetric Hilbert domain (a generalization of a bounded symmetric domain in  $\mathbb{C}^n$ ), then, up to representations vanishing on  $\mathfrak{p}$ , the irreducible semibounded representations of  $G$  are determined in [35, Thm. 8.3].

Here it is natural to assume that  $G$  is the universal central extension of the connected automorphism group of  $\mathcal{D}$ . Then  $K$  is a product of at most three factors isomorphic to  $\mathbb{R}$ , an infinite dimensional group  $U(\mathcal{H})_n$ , or to a simply connected covering group  $\widetilde{U}_n(\mathbb{C})$ . Therefore the separable irreducible bounded representations of  $K$  are well-known (cf. Section 5), so that the main point is to obtain the inequalities characterizing holomorphic inducibility. It is remarkable that, in all cases, the central extension is needed to have semibounded representations that are non-trivial on  $\mathfrak{p}$ .

(b) A concrete example of a full hermitian group is the universal central extension  $G = \widehat{\mathrm{Sp}}_{\mathrm{res}}(\mathcal{H})$  of the restricted symplectic group of a complex Hilbert space with respect to the symplectic form  $\omega(v, w) := \mathrm{Im}\langle v, w \rangle$ :

$$\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}) := \{g \in \mathrm{Sp}(\mathcal{H}, \omega) : \|gg^\top - \mathbf{1}\|_2 < \infty\}.$$

Here  $K \cong \mathbb{R} \times U(\mathcal{H})$ . An important example of a semibounded representation is the metaplectic representation of  $\widehat{\mathrm{Sp}}_{\mathrm{res}}(\mathcal{H})$  on the bosonic Fock space  $S(\mathcal{H}) = \bigoplus_{n \in \mathbb{N}_0} S^n(\mathcal{H})$  ([55]).

(c) Another example of a full irreducible hermitian Lie group is the conformal group  $G = O(\mathbb{R}^2, \mathcal{H})$  of an infinite dimensional Minkowski space. Neither  $G$  nor any of its covering groups have non-trivial semibounded representations ([35, Thm. 8.5]).

**Example 8.4.** (a) For the hermitian groups  $G$  for which the Cartan dual Lie algebra  $\mathfrak{g}^c = \mathfrak{k} \oplus i\mathfrak{p}$  is of the type considered in Example 8.3, i.e.,  $G/K$  is dual to a symmetric Hilbert domain, the classification of the semibounded representations is quite easy to describe because in this case a bounded representation  $(\rho, V)$  of  $K$  is holomorphically inducible if and only if  $\rho$  is *anti-dominant* in the sense that

$$\mathrm{d}\rho([z^*, z]) \geq 0 \quad \text{for} \quad z \in \mathfrak{p}^+ \subseteq \mathfrak{g}_{\mathbb{C}}$$

([35, Thm. 9.1]).

(b) A concrete example is the restricted unitary group of a Hilbert space  $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ :

$$G = U_{\mathrm{res}}(\mathcal{H}, \mathcal{H}_+) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(\mathcal{H}) : \|b\|_2, \|c\|_2 < \infty \right\},$$

for which  $K \cong U(\mathcal{H}_+) \times U(\mathcal{H}_-)$  and  $G/K$  is the *restricted Grassmannian* (cf. [48], [28]).

## 9. Loop groups (Affine Kac–Moody groups)

A *Hilbert–Lie algebra* is a real Lie algebra  $\mathfrak{k}$  carrying the structure of a real Hilbert space, such that the scalar product  $(\cdot, \cdot)$  is invariant under the adjoint action, i.e.,

$$([x, y], z) = (x, [y, z]) \quad \text{for} \quad x, y, z \in \mathfrak{k}.$$

In finite dimensions, these are precisely the compact Lie algebras. In infinite dimensions,  $\mathfrak{k}$  is a direct sum of an abelian ideal and simple Hilbert–Lie algebras which are isomorphic to  $\mathfrak{u}_2(\mathcal{H})$  (the skew-hermitian Hilbert–Schmidt operators) on an infinite dimensional real, complex or quaternionic Hilbert space  $\mathcal{H}$  ([51]).

Let  $K$  be a simply connected Lie group for which  $\mathfrak{k}$  is a simple Hilbert–Lie algebra and  $\varphi \in \text{Aut}(K)$  an automorphism with  $\varphi^N = \text{id}_K$ . Then the *twisted loop group*

$$\mathcal{L}_\varphi(K) := \left\{ f \in C^\infty(\mathbb{R}, K) : (\forall t \in \mathbb{R}) \, f\left(t + \frac{2\pi}{N}\right) = \varphi^{-1}(f(t)) \right\}$$

is a Fréchet–Lie group with Lie algebra

$$\mathcal{L}_\varphi(\mathfrak{k}) := \left\{ \xi \in C^\infty(\mathbb{R}, \mathfrak{k}) : (\forall t \in \mathbb{R}) \, \xi\left(t + \frac{2\pi}{N}\right) = \mathbf{L}(\varphi)^{-1}(\xi(t)) \right\}.$$

For  $\varphi = \text{id}_K$ , we obtain the loop group  $\mathcal{L}(K) = C^\infty(\mathbb{S}^1, K)$  (see [48] for finite dimensional  $K$  and [37] for the infinite dimensional case).

The subgroup  $\mathcal{L}_\varphi(K) \subseteq C^\infty(\mathbb{R}, K)$  is translation invariant, so that we obtain by  $\alpha_s(f)(t) := f(t+s)$  a smooth action of the circle group  $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$  on  $\mathcal{L}_\varphi(K)$ . The cocycle

$$\omega(\xi, \eta) := \int_0^{2\pi} \langle \xi'(t), \eta(t) \rangle dt$$

defines a central extension

$$\tilde{\mathcal{L}}_\varphi(\mathfrak{k}) := \mathbb{R} \oplus_\omega \mathcal{L}_\varphi(\mathfrak{k}),$$

which leads by  $D(z, \xi) := (0, \xi')$  to the double extension

$$\mathfrak{g} := \widehat{\mathcal{L}}_\varphi(\mathfrak{k}) := (\mathbb{R} \oplus_\omega \mathcal{L}_\varphi(\mathfrak{k})) \rtimes_D \mathbb{R}.$$

We put  $\mathbf{d} := (0, 0, -1) \in \mathfrak{g}$ .

To formulate the main results of [37], we first recall that one has a complete classification of the twisted loop groups for infinite dimensional  $K$ . There are four classes of loop algebras: the untwisted loop algebras  $\mathcal{L}(\mathfrak{u}_2(\mathcal{H}))$ , where  $\mathcal{H}$  is an infinite dimensional real, complex or quaternionic Hilbert space, and a twisted type  $\mathcal{L}_\varphi(\mathfrak{u}_2(\mathcal{H}))$ , where  $\mathcal{H}$  is a complex Hilbert space and  $\varphi(x) = \sigma x \sigma$  holds for an antilinear isometric involution  $\sigma: \mathcal{H} \rightarrow \mathcal{H}$  (this corresponds to complex conjugation of the corresponding matrices). We call  $\varphi$  of *standard type*<sup>11</sup> if either  $\varphi = \text{id}_K$ , or

- $\mathbb{K} = \mathbb{R}$  and  $\varphi(g) := rgr^{-1}$ , where  $r$  is the orthogonal reflection in a hyperplane,
- $\mathbb{K} = \mathbb{C}$ ,  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0$ ,  $\sigma_0$  is a conjugation on  $\mathcal{H}_0$ ,  $\sigma(x, y) := (\sigma_0(x), \sigma_0(y))$  on  $\mathcal{H}$ , and

$$\varphi(g) = S\sigma g \sigma S^{-1} \quad \text{for} \quad S = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}, \quad g \in K = \text{U}_2(\mathcal{H}).$$

<sup>11</sup>This terminology stems from the fact that the standard type automorphisms naturally realize the 7 types of locally affine root systems  $A_J^{(1)}, B_J^{(1)}, C_J^{(1)}, D_J^{(1)}, B_J^{(2)}, C_J^{(2)}$  and  $BC_J^{(2)}$ .



- $\mathbb{K} = \mathbb{C}$ ,  $\mathcal{H} = \mathcal{H}_0 \oplus \mathbb{C} \oplus \mathcal{H}_0$ ,  $\sigma(x, y, z) := (\sigma_0(x), \overline{y}, \sigma_0(z))$ , and

$$\varphi(g) = S\sigma g\sigma S^{-1} \quad \text{for} \quad S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad g \in K = \mathrm{U}_2(\mathcal{H}).$$

In [37], the classification of the semibounded unitary representations of the 1-connected Lie group  $G := \widehat{\mathcal{L}}_\varphi(K)$  corresponding to the double extensions  $\mathfrak{g} = \widehat{\mathcal{L}}_\varphi(\mathfrak{k})$  is obtained if  $\varphi$  is of standard type. For a reduction of the general case to this one, we refer to [27]. The first major step is to show that, for an irreducible semi-bounded representation  $(\pi, \mathcal{H})$ , the operator  $-i\mathbf{d}\pi(\mathbf{d})$  is either bounded from below (positive energy representations) or from above (negative energy representations). Up to passing to the dual representation, we may therefore assume that we are in the first case. Then the minimal spectral value of  $-i\mathbf{d}\pi(\mathbf{d})$  turns out to be an eigenvalue and the centralizer  $H := Z_G(\mathbf{d})$  of  $\mathbf{d}$  in  $G$  acts on the corresponding eigenspace, which leads to a bounded irreducible representation  $(\rho, V)$  of  $H$  from which  $(\pi, \mathcal{H})$  is obtained by holomorphic induction. Since an explicit classification of the bounded irreducible representations of the groups  $Z_G(\mathbf{d})_0$  is available from [29, 35] in terms of  $\mathcal{W}$ -orbits of extremal weights, it remains to characterize those weights  $\lambda$  for which the corresponding representation  $(\rho_\lambda, V_\lambda)$  is holomorphically inducible. This is achieved in [37, Thm. 5.10]. It is equivalent to  $\lambda$  being  $d$ -minimal, and the final step consists in showing that the irreducible  $G$ -representation  $(\pi_\lambda, \mathcal{H}_\lambda)$  corresponding to a  $d$ -minimal weight is actually semibounded ([37, Thm. 6.1]).

For untwisted loop groups  $\widehat{\mathcal{L}}(K)$  and compact groups  $K$ , the corresponding class of representations is well-known from the context of affine Kac–Moody algebras (cf. [48]). In this context one thus obtains the class of positive energy representations, but for infinite dimensional  $K$  the positive energy condition is too weak to make holomorphic induction work (cf. [39]).

## 10. The diffeomorphism group of the circle

For the group  $G := \mathrm{Diff}(\mathbb{S}^1)_+$  of orientation preserving diffeomorphisms of the circle, the Lie algebra is the space

$$\mathfrak{g} = \mathcal{V}(\mathbb{S}^1) = C^\infty(\mathbb{S}^1) \frac{\partial}{\partial \theta}$$

of smooth vector fields on  $\mathbb{S}^1$ . Let  $\mathbf{d} := \partial_\theta$ . We say that a unitary representation  $(\pi, \mathcal{H})$  of  $G$  is a *positive energy representation* if  $-i\mathbf{d}\pi(\mathbf{d}) \geq 0$ . A unitary representation  $(\pi, \mathcal{H})$  of  $G$  is semibounded if and only if either  $\pi$  or  $\pi^*$  is of positive energy [34, Thm. 8.3], but unfortunately all these representations are trivial [34, Thm. 8.7] (see also [16] for the algebraic context).

This is the same phenomenon that we already observed for hermitian groups in Section 8 (see also Remark 2.2(b)).

Up to isomorphism,  $\mathcal{V}(\mathbb{S}^1)$  has a unique central extension

$$\mathfrak{vir} = \mathbb{R}c \oplus_{\omega} \mathcal{V}(\mathbb{S}^1), \quad \omega(f, g) = \int_{\mathbb{S}^1} f'g'' - f''g'$$

called the *Virasoro algebra*. It integrates to a central Lie group extension

$$\mathbf{1} \rightarrow \mathbb{R} \times \mathbb{Z} \rightarrow \text{Vir} \rightarrow \text{Diff}(\mathbb{S}^1)_+ \rightarrow \mathbf{1},$$

called the *Virasoro group*. If  $\widehat{G} = \text{Vir}$ , we write  $\widehat{H} \cong \mathbb{R}^2$  for the inverse image of the subgroup  $H \cong \mathbb{T}$  of rigid rotations in  $G$ . Identifying  $\mathbf{d}$  with the corresponding element  $(0, \mathbf{d})$  in  $\mathfrak{vir}$ , we define positive energy representations as above and, using the classification of open invariant cones in  $\mathfrak{vir}$ , one shows that semiboundedness is equivalent to  $\pi$  or  $\pi^*$  being of positive energy [34, Thm .8.15, Cor. 8.16].

To classify semibounded representations, we can not directly use the method of holomorphic induction in the form described in Section 7 because  $\text{Vir}$  is neither an analytic Lie group nor locally exponential [31]. However, the manifold  $\widehat{G}/\widehat{H} \cong G/H$  carries the structure of a complex manifold on which  $G$  acts smoothly by holomorphic maps [24]. In [40] we show that this fact, together with some refinements due to L. Lempert [25] to make holomorphic induction work for the passage from bounded representations of  $\widehat{H}$  to  $\widehat{G} = \text{Vir}$ . In particular, we show that the irreducible semibounded representations of positive energy are in one-to-one correspondence with the unitary highest weight representations, which have been classified in the 1980s ([15], [11], [17, 18]).

## 11. Semiboundedness for solvable groups

In Sections 8 to 10, we discussed semibounded representations for very specific classes of groups. The following theorem shows that there are structural obstructions for the existence of semibounded representations.

**Theorem 11.1.** ([43]) *If  $G$  is a connected Lie group and either nilpotent or 2-step solvable (the commutator group is abelian), then all semibounded representations factor through the abelian group  $G/(G, G)$ .*

This theorem suggests to take a look at 3-step solvable groups. Typical examples arise as follows. Let  $(V, \omega)$  be locally convex symplectic vector space ( $\omega$  is assumed to be continuous and non-degenerate). Then the Heisenberg group

$$\text{Heis}(V, \omega) = \mathbb{R} \times V, \quad (z, v)(z', v') := (z + z' + \tfrac{1}{2}\omega(v, v'), v + v')$$

is a Lie group. We consider a homomorphism  $\alpha: \mathbb{R} \rightarrow \text{Sp}(V, \omega)$  defining a smooth action of  $\mathbb{R}$  on  $V$  and write  $D := \alpha'(0) \in \mathfrak{sp}(V, \omega)$  for its infinitesimal generator. Then the (*generalized*) *oscillator group*

$$G = \text{Heis}(V, \omega) \rtimes_{\alpha} \mathbb{R}, \quad (h, t)(h', t') = (h\alpha_t(h'), t + t'), \quad \alpha_t(z, v) = (z, \alpha_t(v))$$

is a 3-step solvable Lie group.

In [57], C. Zellner shows that the classification of semibounded representations of such groups can be completely reduced to the case where  $V$  is the Fréchet space of smooth vectors for a unitary one-parameter group  $U_t = e^{itH}$  with  $H \geq 0$  and  $\omega(v, w) = \text{Im}\langle v, w \rangle$ . Again, every semibounded representation  $(\pi, \mathcal{H})$  either satisfies the positive energy condition  $\inf \text{Spec}(-i\mathbf{d}\pi(H)) > -\infty$  or its dual does.

**Theorem 11.2** (Uniqueness Theorem, [57, 58]). *If  $\text{Spec}(H) \subseteq [\varepsilon, \infty[$  for some  $\varepsilon > 0$ , then all irreducible semibounded positive energy representations of  $G$  are Fock representations (classified by the lowest eigenvalue of  $-i\mathbf{d}\pi(H)$ ) and every positive energy representation is type I and a direct integral of Fock representations.*

The situation becomes more complicated if  $\inf(\text{Spec}(H)) = 0$ . Then the semibounded representation theory of  $G$  is no longer type I [57]. More precisely, if  $V$  is countably dimensional, then all representations of  $\text{Heis}(V, \omega)$  extend to semibounded representations of some oscillator group  $G$ , so that the classification problem (for all oscillator groups) is equivalent to the classification of the irreducible representations of the Canonical Commutation Relations (CCR), which is a “wild” problem.

## 12. Conclusion and perspectives

We conclude this article with a brief discussion of further developments and of directions in which the theory of semibounded representations can move from here.

**Non-type I representations:** We have seen above that the semibounded representations form a sector of the unitary representation theory of infinite dimensional Lie groups for which the full machinery of  $C^*$ -algebras is available on the abstract level. Presently, the theory is rather well developed for many important classes of Lie groups for which the corresponding representation theory is of type I. Beyond type I representations, we have seen how to reduce the bounded representation theory of gauge groups to the representation theory of UHF algebras. However, for generalized oscillator groups similar results are still missing (Section 11). Here, and in many other cases, one would like to see a class of  $C^*$ -algebras, generalizing UHF algebras, that can host bounded and semibounded representations which are not of type I. Natural candidates for such algebras show up in the context of positive energy representations of gauge groups and have to be explored further [21].

**Positive energy representations:** Another important problem is a better understanding of the relations between the positive energy condition  $-i\mathbf{d}\pi(\mathbf{d}) \geq 0$  for a specific element  $\mathbf{d} \in \mathfrak{g}$  and the semiboundedness condition. We have seen above in Sections 8-10, for a well-chosen  $\mathbf{d}$ , semiboundedness of  $\pi$  often becomes equivalent to either  $\pi$  or  $\pi^*$  being of positive energy. This phenomenon also appears in [21], but in [39] and the closely related [26], the situation is more complicated. To understand these issues, we need a better theory of open invariant cones in infinite dimensional Lie algebras; see [34] for some first steps in this direction.

**Holomorphic induction from unbounded representations:** In Section 7 we have seen how to obtain semibounded representations of a Lie group  $G$  by holomorphic induction from bounded representations of a subgroup  $H$ . This method is strong enough to cover a large variety of cases, such as the ones discussed in Sections 8-10. However, generalizations are needed for other classes of groups because there are natural situations, where the representation of  $H$  on a subspace  $V$  generated by its intersection with  $(\mathcal{H}^\infty)^{\mathfrak{p}^-}$  is not bounded. In this case one has to work with weaker structures on the holomorphic bundles by either using the dense subspace  $V^\infty$  as fibers of the bundle or by relaxing the smoothness of the structure on the bundle.

**Unitary representations of Lie supergroups:** In this survey we did not touch representations of Lie supergroups, although they are closely related to semibounded representations. In the unitary representation theory of Lie supergroups, the most important class of representations are those for which the corresponding unitary representation of the even part  $G$  with Lie algebra  $\mathfrak{g}_{\overline{0}}$  is semibounded because we always have  $-i\mathrm{d}\pi([x, x]) \geq 0$  for any odd element  $x$  in the Lie superalgebra. This has the consequence that the method of smoothing operators and the theory of semibounded representations applies particularly nicely to supergroups. We refer to [41] for the construction of full “host algebras” for finite dimensional Lie supergroups. We expect similar constructions to work for large classes of infinite dimensional Lie supergroups as well.

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### 13. References

- [1] B. Bakalov, N. M. Nikolov, K.-H. Rehren, and I. Todorov, Unitary positive energy representations of scalar bilocal quantum fields, *Comm. Math. Phys.* **271:1** (2007), 223–246
- [2] D. Beltiță, and K.-H. Neeb, A non-smooth continuous unitary representation of a Banach–Lie group, *J. Lie Theory* **18** (2008), 933–936
- [3] —, Schur–Weyl Theory for  $C^*$ -algebras, *Math. Nachrichten* **285:10** (2012), 1170–1198
- [4] —, Nonlinear completely positive maps and dilation theory for real involutive algebras, *Integral Equations Operator Theory*, to appear; arXiv: 1411.6398 [math.OA]

- [5] H. J. Borchers, *Translation group and particle representations in quantum field theory*, Lecture Notes in Physics, Springer, 1996
- [6] R. Boyer, Representations of the Hilbert Lie group  $U(\mathcal{H})_2$ , *Duke Math. J.* **47**(1980), 325–344.
- [7] A. L. Carey, Infinite Dimensional Groups and Quantum Field Theory, *Act. Appl. Math.* **1** (1983), 321–333
- [8] A. L. Carey and S. N. M. Ruijsenaars, On fermion gauge groups, current algebras, and Kac–Moody algebras, *Acta Appl. Math.* **10** (1987), 1–86
- [9] J. Dixmier,  *$C^*$ -algebras*, North Holland Publishing Company, Amsterdam, New York, Oxford, 1977
- [10] C. Fewster, and S. Hollands, Quantum energy inequalities in two-dimensional conformal field theory, *Rev. Math. Phys.* **17:5** (2005), 577–612
- [11] D. Friedan, Z. Qiu and S. Shenker, Details of the non-unitarity proof for highest weight representations of the Virasoro algebra, *Comm. Math. Phys.* **107** (1986), 535–542
- [12] L. Gårding, Note on continuous representations of Lie groups, *Proc. Nat. Acad. Sci. U.S.A.* **33** (1947), 331–332
- [13] I. M. Gel’fand and N. Ya Vilenkin, *Generalized Functions. Vol. 4: Applications of Harmonic Analysis*, Translated by Amiel Feinstein, Academic Press, New York, London, 1964
- [14] J. Glimm, On a certain class of operator algebras, *Transactions of the Amer. Math. Soc.* **40** (1960), 318–340
- [15] P. Goddard, A. Kent and D. Olive, Unitary representations of the Virasoro and super-Virasoro algebras, *Commun. Math. Phys.* **103** (1986), 105–119
- [16] P. Goddard and D. Olive, Kac–Moody and Virasoro algebras in relation to quantum physics, *Internat. J. Mod. Phys. A* **1** (1986), 303–414
- [17] R. Goodman and N. R. Wallach, Structure and unitary cocycle representations of loop groups and the group of diffeomorphisms of the circle, *J. reine ang. Math.* **347** (1984), 69–133
- [18] —, Projective unitary positive energy representations of  $\text{Diff}(\mathbb{S}^1)$ , *J. Funct. Anal.* **63** (1985), 299–312
- [19] H. Grundling, Generalizing group algebras, *J. London Math. Soc.* **72** (2005), 742–762
- [20] B. Janssens and K.-H. Neeb, Norm continuous unitary representations of Lie algebras of smooth sections, *Internat. Math. Research Notices*, 2014, doi:10.1093/imrn/rnu231; arXiv:1302.2535 [math.RT]
- [21] —, Positive energy representations of gauge groups, in preparation
- [22] M. Keyl, J. Kiukas, and R. F. Werner, Schwartz operators, Preprint, arXiv:1503.04086 [math-ph]
- [23] A. A. Kirillov, Representation of the infinite-dimensional unitary group, *Dokl. Akad. Nauk. SSSR* **212**(1973), 288–290
- [24] A. A. Kirillov and D. V. Yuriev, Representations of the Virasoro algebra by the orbit method, *J. Geom. Phys.* **5:3** (1988), 351–363

- [25] L. Lempert, The Virasoro group as a complex manifold, *Math. Res. Lett.* **2:4** (1995), 479–495
- [26] T. Marquis and K.-H. Neeb, Positive energy representations for locally finite split Lie algebras, Preprint, arXiv:RT.1507.06077
- [27] —, Isomorphisms of twisted Hilbert–Loop algebras, in preparation
- [28] J. Mickelsson, *Current Algebras and Groups*, Plenum Press, New York, 1989
- [29] K.-H. Neeb, Holomorphic highest weight representations of infinite dimensional complex classical groups, *J. Reine Angew. Math.* **497** (1998), 171–222
- [30] —, *Holomorphy and Convexity in Lie Theory*, Expositions in Mathematics **28**, de Gruyter Verlag, Berlin, 2000
- [31] —, Towards a Lie theory of locally convex groups, *Jap. J. Math. 3rd ser.* **1:2** (2006), 291–468
- [32] —, A complex semigroup approach to group algebras of infinite dimensional Lie groups, *Semigroup Forum* **77** (2008), 5–35
- [33] —, Semibounded unitary representations of infinite dimensional Lie groups, in *Infinite Dimensional Harmonic Analysis IV*, Eds. J. Hilgert et al, World Scientific, 2009; 209–222
- [34] —, Semibounded representations and invariant cones in infinite dimensional Lie algebras, *Confluentes Math.* **2:1** (2010), 37–134
- [35] —, Semibounded representations of hermitian Lie groups, *Travaux mathématiques* **21** (2012), 29–109
- [36] —, Holomorphic realization of unitary representations of Banach–Lie groups, in *Lie Groups: Structure, Actions, and Representations—In Honor of Joseph A. Wolf on the Occasion of his 75th Birthday*, Huckleberry, A., Penkov, I., Zuckerman, G. (Eds.), Progress in Mathematics **306**, 2013; 185–223
- [37] —, Semibounded unitary representations of double extensions of Hilbert–Loop groups, *Ann. Inst. Fourier*, **64:5** (2014), 1823–1892; arXiv:1104.2234 [math.RT]
- [38] —, Unitary representations of unitary groups, in *Lie Theory Workshops*, Eds. G. Mason, I. Penkov, J. Wolf, Developments in Math. **37**, Springer, 2014, 197–243
- [39] —, Projective semibounded representations of doubly extended Hilbert–Lie groups, in preparation
- [40] K.-H. Neeb and H. Salmasian, Classification of positive energy representations of the Virasoro group, *Internat. Math. Research Notices*, 2014; rnu197, 37 pages; doi:10.1093/imrn/rnu197
- [41] —, Crossed product algebras and direct integral decomposition for Lie supergroups, *Pacific J. Math.*, to appear; arxiv:1506.01558 [math.RT]
- [42] K.-H. Neeb, H. Salmasian and C. Zellner, Smoothing operators and  $C^*$ -algebras for infinite dimensional Lie groups, arXiv:1506.01558 [math.RT]
- [43] K. H. Neeb and C. Zellner, Oscillator algebras with semi-equicontinuous coadjoint orbits, *Differential Geometry and its Applications* **31:2** (2013), 268–283
- [44] E. Nelson, Analytic vectors, *Ann. of Math.* **70** (1969), 572–615
- [45] G. I. Olshanski, Unitary representations of the infinite-dimensional classical groups  $U(p, \infty)$ ,  $SO_0(p, \infty)$ ,  $Sp(p, \infty)$ , and of the corresponding motion groups, *Functional Anal. Appl.* **12:3** (1978), 185–195

- [46] D. Pickrell, The separable representations of  $U(\mathcal{H})$ , *Proc. of the Amer. Math. Soc.* **102** (1988), 416–420
- [47] R. T. Powers, Representations of uniformly hyperfinite algebras and their associated von Neumann algebras, *Annals of Math.* **86** (1967), 138–171
- [48] A. Pressley and G. Segal, *Loop Groups*, Oxford University Press, Oxford, 1986
- [49] S. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, New York, 1973
- [50] S. Sakai, On a characterization of type I  $C^*$ -algebras, *Bull. Amer. Math. Soc.* **72** (1966), 508–512
- [51] J. R. Schue, Hilbert space methods in the theory of Lie algebras, *Transactions of the Amer. Math. Soc.* **95** (1960), 69–80
- [52] G. Segal, Unitary representations of some infinite-dimensional groups, *Comm. Math. Phys.* **80** (1981), 301–342
- [53] I. E. Segal, The structure of a class of representations of the unitary group on a Hilbert space, *Proc. Amer. Math. Soc.* **81** (1957), 197–203
- [54] —, Positive-energy particle models with mass splitting, *Proc. Nat. Acad. Sci. U.S.A.* **57** (1967), 194–197
- [55] —, The complex wave representation of the free boson field, in *Topics in Funct. Anal.*, Adv. in Math. Suppl. Studies **3** (1978), 321–343
- [56] Y. Yamasaki, *Measures on Infinite Dimensional Spaces*, Series in Pure Math. **5**, World Scientific, Singapore, 1985
- [57] C. Zellner, *Semibounded unitary representations of oscillator groups*, PhD Thesis, Friedrich–Alexander University Erlangen–Nuremberg, 2014
- [58] —, Complex semigroups for oscillator groups, arXiv:1506.06240 [math.FA]

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